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A NEW SOLUTION OF KIRCHHOFF'S EQUATIONS OF THE PROBLEM OF THE MOTION OF A GYROSTAT ACTED UPON BY POTENTIAL AND GYROSCOPIC FORCES†

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The conditions for the existence of special solutions, for which the components of the angular momentum vector are the superposition of linear and linear-fractional functions are considered for Kirchhoff's differential equations, which describe the motion of a gyrostat under potential and gyroscopic forces.

A noteworthy feature of Kirchhoff class equations [1] is the fact that, by a non-degenerate linear transformation of the main variables of the problem, they can be converted into the equations of motion of a charged and magnetized gyrostat in Newtonian, electric and magnetic fields. This is a hydrodynamic analogy for special cases, pointed out by Steklov [2] and Kharlamov [3] and obtained in completed form in [4, 5]. There are many approaches [6–9] to the investigation of the properties of integral manifolds of Kirchhoff's equations. In view of the non-integrability of these equations in quadratures [7], an approach based on constructing special solutions using the method of invariant relations [10] is important.

In this paper we construct a new solution of these equations for the case when the characteristic matrices occurring on the right-hand side of Kirchhoff's equations are diagonal, while the vectors of the generalized centre of mass and of the gyrostatic moment are directed along the principal axis. It possesses a new structure of the auxiliary invariant relations, which give the components of the angular momentum vector in terms of the components of the unit vector of the axis of symmetry of the Newtonian, electric and magnetic fields.

1. FORMULATION OF THE PROBLEM. FORM OF THE SOLUTION

Consider the problem of the motion of a gyrostat with a fixed point under potential and gyroscopic forces, which is described by Kirchhoff class equations [4, 5, 9]

$$\dot{\mathbf{x}} = (\mathbf{x}+\boldsymbol{\lambda})\times a\mathbf{x} + a\mathbf{x}\times B\mathbf{v} + \mathbf{s}\times\mathbf{v} + \mathbf{v}\times C\mathbf{v} \tag{1.1}$$

$$\dot{\mathbf{v}} = \mathbf{v}\times a\mathbf{x} \tag{1.2}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ is the angular momentum vector of the gyrostat, $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$ is the unit vector of the axis of symmetry of the force field, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ is the gyrostatic moment, characterizing the motion of supported bodies, $\mathbf{s} = (s_1, s_2, s_3)$ is a vector, codirectional with the vector of the generalized centre of mass of the gyrostat, $a = (a_{ij})$ is the gyration tensor of the gyrostat, constructed at a fixed point, and $B = (B_{ij})$, $C = (C_{ij})$ are third-order constant symmetrical matrices; the dot above the variables $\mathbf{x}$ and $\mathbf{v}$ denotes a derivative with respect to time t.

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Equations (1.1) and (1.2) have the following first integrals

$$\mathbf{x}\cdot a\mathbf{x}-2(\mathbf{s}\cdot\mathbf{v})+(C\mathbf{v}\cdot\mathbf{v}) = 2E,\quad \mathbf{v}\cdot\mathbf{v} = 1$$
$$(\mathbf{x}+\boldsymbol{\lambda})\cdot\mathbf{v}-\frac{1}{2}(B\mathbf{v}\cdot\mathbf{v}) = k \tag{1.3}$$

Here E and k are arbitrary constants.

Suppose the matrices a, B and C have a diagonal structure with elements a_i, and B_i and C_i ($i = 1, 2, 3$), while the vectors $\mathbf{s}$ and $\boldsymbol{\lambda}$ are directed along the first principal axis of the gyration ellipsoid: $\mathbf{s} = (s_1, 0, 0)$, $\boldsymbol{\lambda} = (\lambda_1, 0, 0)$.

We will investigate the solutions of Eqs (1.1) and (1.2), which are characterized by three invariant relations [11, 12]

$$x_1 = \varphi_1(\nu_1),\quad x_2 = \nu_2\varphi_2(\nu_1),\quad x_3 = \nu_3\varphi_3(\nu_1) \tag{1.4}$$

Then, using the geometrical integral from (1.3), the vector equations (1.1) and (1.2) can be converted to the following five equations

$$\psi'(\nu_1) = 2\frac{a_1\varphi_1(\nu_1)-a_3\nu_1\varphi_3(\nu_1)}{a_3\varphi_3(\nu_1)-a_2\varphi_2(\nu_1)} \tag{1.5}$$

$$(a_3\varphi_3(\nu_1)-a_2\varphi_2(\nu_1))\varphi_1'(\nu_1) =$$
$$= (a_3-a_2)\varphi_2(\nu_1)\varphi_3(\nu_1)+a_2B_3\varphi_2(\nu_1)-a_3B_2\varphi_3(\nu_1)+C_3-C_2 \tag{1.6}$$

$$\psi(\nu_1)(a_3\varphi_3(\nu_1)-a_2\varphi_2(\nu_1))\varphi_2'(\nu_1) =$$
$$= \varphi_2(\nu_1)(a_3\nu_1\varphi_3(\nu_1)-a_1\varphi_1(\nu_1))+(a_1-a_3)\varphi_1(\nu_1)\varphi_3(\nu_1)- \tag{1.7}$$
$$-a_3\lambda_1\varphi_3(\nu_1)+a_3B_1\nu_1\varphi_3(\nu_1)-a_1B_3\varphi_1(\nu_1)-s_1+(C_1-C_3)\nu_1$$

$$(1-\nu_1^2-\psi(\nu_1))(a_3\varphi_3(\nu_1)-a_2\varphi_2(\nu_1))\varphi_3'(\nu_1) =$$
$$= \varphi_3(\nu_1)(a_1\varphi_1(\nu_1)-a_2\nu_1\varphi_2(\nu_1))+(a_2-a_1)\varphi_1(\nu_1)\varphi_2(\nu_1)+ \tag{1.8}$$
$$+a_2\lambda_1\varphi_2(\nu_1)-a_2B_1\nu_1\varphi_2(\nu_1)+a_1B_2\varphi_1(\nu_1)+s_1+(C_2-C_1)\nu_1$$

$$\dot\nu_1 = (a_3\varphi_3(\nu_1)-a_2\varphi_2(\nu_1))\sqrt{\psi(\nu_1)(1-\nu_1^2-\psi(\nu_1))} \tag{1.9}$$

where $\psi(\nu_1) = \nu_2^2(\nu_1)$; the prime denotes a derivative with respect to the auxiliary variable ν_1.

If a certain solution $\psi = \psi(\nu_1)$, $\varphi_i(\nu_1)$ ($i = 1, 2, 3$) of Eqs (1.5)–(1.8) is obtained, we can determine the relation $\nu_1 = \nu_1(t)$ from Eqs (1.9). The components of the angular momentum vector are then obtained from relations (1.4), where

$$\nu_2(\nu_1) = \sqrt{\psi(\nu_1)},\quad \nu_3(\nu_1) = \sqrt{1-\nu_1^2-\psi^2(\nu_1)} \tag{1.10}$$

The integrals of the energy and the angular momentum from system (1.3) based on invariant relations (1.4) are as follows:

$$\nu_1\varphi_1(\nu_1)+\psi(\nu_1)\varphi_2(\nu_1)+(1-\nu_1^2-\psi(\nu_1))\varphi_3(\nu_1) = \frac{1}{2}(b_0\psi+n_0+n_1\nu_1+n_2\nu_1^2)$$
$$a_1\varphi_1^2(\nu_1)+a_2\psi(\nu_1)\varphi_2^2(\nu_1)+a_3(1-\nu_1^2-\psi(\nu_1))\varphi_3^2(\nu_1) = c_0\psi+m_0+m_1\nu_1+m_2\nu_1^2 \tag{1.11}$$

where

$$b_0 = B_2-B_3,\quad n_1 = -2\lambda_1,\quad n_2 = B_1-B_3$$
$$c_0 = C_3-C_2,\quad m_1 = 2s_1,\quad m_2 = C_3-C_1 \tag{1.12}$$

and n_0 and m_0 are arbitrary constants, introduced instead of E and k.

When investigating the conditions for the existence of invariant relations (1.4) for Eqs (1.1) and (1.2) with integrals (1.11), we considered the case [12] when the functions $\psi(\nu_1)$ and $\varphi_i(\nu_1)$ in Eqs (1.5)–(1.9) are polynomials in the variable ν_1. Hence, it is of interest to investigate the solutions of Eqs (1.5)–(1.9) in a more general form.

We will specify the solution of Eqs (1.5)–(1.9) using the following in invariant relations

$$\begin{aligned}\psi(\nu_1) &= \alpha_2\nu_1^2+\alpha_1\nu_1+\alpha_0\\ \varphi_2(\nu_1) &= \gamma_0+\frac{\gamma_1}{\nu_1+\varepsilon_0},\quad \varphi_3(\nu_1) = \frac{1}{a_3}(\mu_0+a_2\varphi_2(\nu_1))\end{aligned} \tag{1.13}$$

where α_2, α_1, α_0, γ_0, γ_1, μ_0 are constants, which depend on the parameters of problem (1.1), (1.2), to be determined. The basis of this approach is not only the more complex structure of the solution compared with that employed previously in [12], but also the fact that, when the third equality of (1.13) is satisfied, Eq. (1.9) takes the form

$$\dot{\nu}_1 = \{(\alpha_2\nu_1^2+\alpha_1\nu_1+\alpha_0)[-(1+\alpha_2)\nu_1^2-\alpha_1\nu_1+(1-\alpha_0)]\}^{1/2} \tag{1.14}$$

i.e. $\nu_1 = \nu_1(t)$ is an elliptic function of time. The latter property is characteristic for the majority of special solutions of the equations of rigid-body dynamics [6].

Note that, on the basis of relations (1.13), the function $\varphi_1(\nu_1)$ can be obtained from Eq. (1.5)

$$\varphi_1(\nu_1) = \frac{1}{2a_1}[\mu_0(2(\alpha_2+1)\nu_1+\alpha_1)+2a_2\nu_1\varphi_2(\nu_1)] \tag{1.15}$$

Equations (1.11), (1.12) give the values of the constants of the first integrals in the solution considered.

2. THE CONDITIONS FOR SOLUTION (1.13)–(1.15) TO EXIST

Using expressions (1.13) and (1.15), Eqs (1.6)–(1.18) can be written as follows ($u = \nu_1 + \varepsilon_0$ is a new variable):

$$\mu_0a_3\varphi_1'(u)+a_2(a_2-a_3)\varphi_2^2(u)-x_0\varphi_2(u)+\sigma_0 = 0 \tag{2.1}$$

$$\begin{aligned}&2\mu_0a_1a_3\psi(u)\varphi_2'(u) =\\ &= 2a_2^2(a_1-a_3)(u-\varepsilon_0)\varphi_2^2(u)+(G_1u+G_0)\varphi_2(u)+D_1u+D_0\end{aligned} \tag{2.2}$$

$$\begin{aligned}&2\mu_0a_1a_2(1-(u-\varepsilon_0)^2-\psi(u))\varphi_2'(u) =\\ &= 2a_2a_3(a_2-a_1)(u-\varepsilon_0)\varphi_2^2(u)+(K_1u+K_0)\varphi_2(u)+M_1u+M_0\end{aligned} \tag{2.3}$$

where

$$\psi(u) = \alpha_2u^2+(\alpha_1-2\varepsilon_0\alpha_2)u+(\alpha_0-\varepsilon_0\alpha_1+\varepsilon_0^2\alpha_2)$$

$$\varphi_2(u) = \gamma_0+\gamma_1u^{-1}$$

$$x_0 = \mu_0(a_3-a_2)+a_2a_3(B_3-B_2),\quad \sigma_0 = a_3(\mu_0B_2+C_2-C_3)$$

$$G_1 = 2[\mu_0\alpha_2(a_1a_2-a_1a_3-a_2a_3)+2\mu_0a_2(a_1-a_3)+a_1a_2a_3(B_1-B_3)]$$

$$\begin{aligned}G_0 &= 2\mu_0\varepsilon_0\alpha_2(-a_1a_2+a_1a_3+a_2a_3)-4\mu_0\varepsilon_0a_2(a_1-a_3)+\\ &+\mu_0\alpha_1(a_1a_2-a_1a_3-a_2a_3)-2\lambda_1a_1a_2a_3-2\varepsilon_0a_1a_2a_3(B_1-B_3)\end{aligned}$$

$$D_1 = 2[\mu_0^2(a_1-a_3)(\alpha_2+1)+\mu_0a_1a_3(B_1-B_3)-\mu_0a_1a_3B_3\alpha_2+a_1a_3(C_1-C_3)] \tag{2.4}$$

$$D_0 = \mu_0^2(a_1-a_3)[\alpha_1-2\varepsilon_0(1+\alpha_2)]-2\mu_0\lambda_1a_1a_3-\mu_0a_1a_3B_3(\alpha_1-2\varepsilon_0\alpha_2)-$$
$$-2\varepsilon_0a_1a_3(C_1-C_3)-2s_1a_1a_3-2\mu_0\varepsilon_0a_1a_3(B_1-B_3)$$

$$K_1 = 2[\mu_0(a_1a_2-a_1a_3+a_2a_3)(\alpha_2+1)+a_1a_2a_3(B_2-B_1)]$$

$$K_0 = \mu_0(a_1a_2-a_1a_3+a_2a_3)[\alpha_1-2\varepsilon_0(\alpha_2+1)]+2\lambda_1a_1a_2a_3-2\varepsilon_0a_1a_2a_3(B_2-B_1)$$

$$M_1 = 2\mu_0a_1(\mu_0+a_3B_2)(\alpha_2+1)+2a_1a_3(C_2-C_1)$$

$$M_0 = \mu_0a_1(\mu_0+a_3B_2)[\alpha_1-2\varepsilon_0(\alpha_2+1)]+2s_1a_1a_3-2\varepsilon_0a_1a_3(C_2-C_1)$$

where, by virtue of expression (1.15), $\varphi_1(u)$ has the form

$$\varphi_1(u) = \frac{1}{2a_1}[\mu_0(\alpha_1-2\varepsilon_0(\alpha_2+1))+2(\gamma_1-\varepsilon_0\gamma_0)a_2+$$
$$+2(\mu_0(\alpha_2+1)+\gamma_0a_2)u-2\varepsilon_0\gamma_1a_2u^{-1}] \tag{2.5}$$

By requiring that the functions $\varphi_2(u) = \gamma_0 + \gamma_1u^{-1}$ and $\varphi_1(u)$ from relation (2.5) should satisfy Eqs (2.1)–(2.3), we obtain the following conditions, connecting the parameters of the solution and the parameters of problem (1.1), (1.2)

$$\varepsilon_0^2 = \frac{a_1^2}{(a_1-a_2)(a_1-a_3)},\quad \gamma_0 = \frac{x_0}{2a_2(a_2-a_3)},\quad \gamma_1 = \frac{\mu_0\varepsilon_0a_3}{a_1(a_3-a_2)} \tag{2.6}$$

$$\alpha_0 = \frac{\varepsilon_0}{a_1^2(a_2-a_3)}[\varepsilon_0a_1^2(a_3-a_2)\alpha_2-a_1^2(a_3-a_2)\alpha_1-\varepsilon_0a_2^2(a_1-a_3)] \tag{2.7}$$

$$\gamma_0^2a_1a_2(a_2-a_3)+\gamma_0(\mu_0a_2a_3-a_1x_0)+\mu_0^2a_3(\alpha_2+1)+\sigma_0a_1 = 0 \tag{2.8}$$

$$2\mu_0a_1a_3(\alpha_1-2\varepsilon_0\alpha_2)-4\varepsilon_0\gamma_0a_2^2(a_1-a_3)+2\gamma_1a_2^2(a_1-a_3)+G_0 = 0 \tag{2.9}$$

$$2\gamma_0^2a_2^2(a_1-a_3)+\gamma_0G_1+D_1 = 0 \tag{2.10}$$

$$2\mu_0\gamma_1a_1a_3\alpha_2-2\varepsilon_0\gamma_0^2a_2^2(a_1-a_3)+4\gamma_0\gamma_1a_2^2(a_1-a_3)+\gamma_0G_0+\gamma_1G_1+D_0 = 0 \tag{2.11}$$

$$2\mu_0a_1a_2[2\varepsilon_0(\alpha_2+1)-\alpha_1]-4\varepsilon_0\gamma_0a_2a_3(a_2-a_1)+2\gamma_1a_2a_3(a_2-a_1)+K_0 = 0 \tag{2.12}$$

$$2\gamma_0^2a_2a_3(a_2-a_1)+\gamma_0K_1+M_1 = 0 \tag{2.13}$$

$$2\mu_0\gamma_1a_1a_2(\alpha_2+1)+2\varepsilon_0\gamma_0^2a_2a_3(a_2-a_1)-4\gamma_0\gamma_1a_2a_3(a_2-a_1)-\gamma_0K_0-\gamma_1K_1-M_0 = 0 \tag{2.14}$$

Relations (2.6) and (2.7) show that the parameter ε_0 in them is expressed in terms of the components of the gyration tensor, the parameters γ_0 and γ_1 are expressed in terms of the components of the gyration tensor and the quantities B_2, B_3 and μ_0, and the parameter α_0 is expressed in terms of the components of the gyration tensor and the quantities α_1 and α_2. It can be shown from relation (2.4) that system of equations (2.8)–(2.14) is linearly dependent and reduces to the system

$$\mu_0(a_2-a_3)[\alpha_2(a_1a_2+a_1a_3-2a_2a_3)-a_2(a_3-a_1)]+$$
$$+a_2a_3[a_1(a_3-a_2)(B_1-B_3)+a_2(a_3-a_1)(B_3-B_2)] = 0 \tag{2.15}$$

$$\lambda_1 = \frac{\mu_0}{2a_1^2a_2a_3(a_3-a_2)}[a_1(a_3-a_2)(a_1a_2+a_1a_3-a_2a_3)\alpha_1-$$
$$-2\varepsilon_0a_2a_3(a_1(a_3-a_2)\alpha_2+a_2(a_3-a_1))] \tag{2.16}$$

$$\begin{aligned}&\mu_0^2(a_2-a_3)(-a_1a_2+a_1a_3+2a_2a_3+4a_2a_3\alpha_2)+\\&+2\mu_0a_2a_3[B_3(a_1a_2-a_1a_3+a_2a_3)-B_2(-a_1a_2+a_1a_3+a_2a_3)]-\\&-a_1a_2^2a_3^2(B_3-B_2)^2+4a_1a_2a_3(a_2-a_3)(C_2-C_3)=0\end{aligned} \tag{2.17}$$

$$\begin{aligned}&x_0^2a_2(a_1-a_3)+2x_0[\mu_0(a_1a_2-a_1a_3-a_2a_3)\alpha_2+2\mu_0a_2(a_1-a_3)+\\&+a_1a_2a_3(B_1-B_3)]+4a_2(a_2-a_3)[\mu_0^2(a_1-a_3)(\alpha_2+1)+\\&+\mu_0a_1a_3(B_1-B_3)-\mu_0a_1a_3B_3\alpha_2+a_1a_3(C_1-C_3)]=0\end{aligned} \tag{2.18}$$

$$\begin{aligned}&2\mu_0\gamma_1a_1a_3\alpha_2-2\varepsilon_0\gamma_0^2a_2^2(a_1-a_3)+4\gamma_0\gamma_1a_2^2(a_1-a_3)+\\&+\gamma_0G_0+\gamma_1G_1+\mu_0^2(a_1-a_3)[\alpha_1-2\varepsilon_0(\alpha_2+1)]-\\&-2\mu_0\lambda_1a_1a_3-\mu_0a_1a_3B_3(\alpha_1-2\varepsilon_0\alpha_2)-2\varepsilon_0a_1a_3(C_1-C_3)-\\&-2\mu_0\varepsilon_0a_1a_3(B_1-B_3)-2s_1a_1a_3=0\end{aligned} \tag{2.19}$$

Hence, Eqs (2.6), (2.7) and (2.15)–(2.19) serve as the conditions for solution (1.13)–(1.15) for Kirchhoff's equations (1.1) and (1.2) to exist.

If we use the fact that, by their mechanical meaning, the quantities λ_1, s_1, C_2-C_3 and C_1-C_3 can take arbitrary values, then, to prove the solvability of Eqs (2.6), (2.7) and (2.15)–(2.19) we can use the semi-inverse method, which enables us to avoid lengthy calculations. Suppose we are given the values of the parameters $a_1, a_2, a_3, B_1, B_2, B_3$, α_2 and α_1. Then, from Eq. (2.15) we can determine the parameter μ_0, and, from Eqs (2.6), the values of the parameters of the solution: ε_0, γ_0 and γ_1. From the results obtained from Eq. (2.7) we can find the parameter α_0, from Eq. (2.16) we can find the parameter λ_1, from Eq. (2.17) we can find the parameter C_2-C_3, from Eq. (2.18) we can find the parameter C_1-C_3, and from Eq. (2.19) we can find the parameter s_1, since these parameters occur linearly in conditions (2.15)–(2.19). It is necessary to take into account here that the values of α_0, α_1 and α_2 obtained must satisfy the conditions for the solution to be real

$$\begin{aligned}\nu_2^2(\nu_1)&=\psi(\nu_1)=\alpha_2\nu_1^2+\alpha_1\nu_1+\alpha_0\geq 0\\\nu_3^2(\nu_1)&=-(1+\alpha_2)\nu_1^2-\alpha_1\nu_1+(1-\alpha_0)\geq 0\end{aligned} \tag{2.20}$$

This can be achieved, for example, by choosing the quantities α_1 and α_2, assuming that the right-hand side Eq. (2.7) is positive and does not exceed unity. The functions $\nu_2^2(\nu_1)$ and $\nu_3^2(\nu_1)$ at the point $\nu_1=0$ are then positive, and, in view of their continuity, a non-empty interval in ν_1 exists, in which conditions (2.20) are satisfied.

Hence, when Eqs (2.6) and (2.7) are satisfied, Eqs (1.1), (1.2) allows of the solution (it is found using formulae (1.4), (1.10) and (1.13)–(1.15))

$$\begin{aligned}&\nu_2(\nu_1)=\sqrt{\alpha_2\nu_1^2+\alpha_1\nu_1+\alpha_0},\quad \nu_3(\nu_1)=\sqrt{-(1+\alpha_2)\nu_1^2-\alpha_1\nu_1+(1-\alpha_0)}\\&x_1(\nu_1)=\frac{1}{2a_1}\left[\mu_0\alpha_1+2(\mu_0(\alpha_2+1)+\gamma_0a_2)\nu_1+\frac{2\gamma_1a_2\nu_1}{\nu_1+\varepsilon_0}\right]\\&x_2(\nu_1)=\nu_2(\nu_1)\left(\gamma_0+\frac{\gamma_1}{\nu_1+\varepsilon_0}\right),\quad x_3(\nu_1)=\frac{\nu_3(\nu_1)}{a_3}\left(\mu_0+\gamma_0a_2+\frac{\gamma_1a_2}{\nu_1+\varepsilon_0}\right)\\&\dot{\nu}_1=\nu_2(\nu_1)\nu_3(\nu_1)\end{aligned} \tag{2.21}$$

The noteworthy property of solution (2.21) is the structure of the functions $x_i(\nu_1)$: they are the superposition of linear and linear-fractional functions of the components of the vector of the axis of symmetry of the force field. Since the constants n_0 and m_0, occurring in relation (1.11), take fixed values, which we will not write here in view of their complexity, the solution (2.21) depends on one arbitrary constant t_0. This constant arises when solving the last equation of system (2.21).

3. NUMERICAL EXAMPLE

We will specify the following values of the components of the gyration tensor: $a_1 = 2a$, $a_2 = 3a$, $a_3 = 5a$, where a is a parameter. From the first equation of system (2.6) we then obtain $\varepsilon_0 = 2\sqrt{3}/3$. Assuming the parameter μ_0 to be free, the following quantities satisfy Eq. (2.15)

$$B_1 = \frac{3\mu_0}{a}, \quad B_2 = \frac{\mu_0}{a}, \quad B_3 = \frac{8\mu_0}{5a}$$

Equation (2.7) is satisfied, for example, under the following conditions

$$\alpha_2 = \frac{21}{4}, \quad \alpha_1 = \frac{25\sqrt{3}}{4}, \quad \alpha_0 = 1$$

From the notation for x_0, from (2.4) we have $x_0 = 11\mu_0 a$.

We obtain the value of λ_1 from (2.16), and the values of γ_0 and γ_1 from the second and third equations of system (2.6), we have

$$\lambda_1 = -\frac{115\sqrt{3}\mu_0}{48a}, \quad \gamma_0 = -\frac{11\mu_0}{12a}, \quad \gamma_1 = \frac{5\sqrt{3}\mu_0}{6a}$$

We return to Eqs (2.17)–(2.19). Substituting the values of the parameters obtained above into these, we obtain

$$C_1 - C_3 = \frac{21\mu_0^2}{5a^3}, \quad C_2 - C_1 = -\frac{191\mu_0^2}{24a^3}, \quad s_1 = -\frac{5\sqrt{3}\mu_0^2}{2a^3}$$

Hence, we have given an example of the solvability of conditions (2.6), (2.7) and (2.15)–(2.19). Solution (2.21) takes the form

$$\nu_2(\nu_1) = \frac{1}{2}\sqrt{21(\nu_1 - \nu_1^{(0)})(\nu_1 - \nu_2^{(0)})}, \quad \nu_3(\nu_1) = \frac{5}{2}\sqrt{-\nu_1(\nu_1 + \sqrt{3})}$$
$$(\nu_1^{(0)} \approx -1.96, \quad \nu_2^{(0)} \approx -0.09)$$
$$x_1 = \frac{\mu_0}{a}\left(\frac{45\sqrt{3}}{16} + \frac{7}{4}\nu_1 - \frac{5\sqrt{3}}{2(\sqrt{3}\nu_1 + 2)}\right)$$
$$x_2 = \frac{\mu_0\nu_2(\nu_1)}{a}\left(-\frac{11}{12} + \frac{5}{2(\sqrt{3}\nu_1 + 2)}\right), \quad x_3 = \frac{\mu_0\nu_3(\nu_1)}{5a}\left(-\frac{7}{4} + \frac{15}{2(\sqrt{3}\nu_1 + 2)}\right)$$
$$\dot{\nu}_1 = \nu_2(\nu_1)\nu_3(\nu_1)$$

The variable ν_1 varies in the range $[\nu_2^{(0)}, 0]$.

An analysis of the literature devoted to constructing general and special solutions of Kirchhoff's equations shows that solution (2.21), according to the conditions imposed on the parameters, cannot be a special case of the solutions of the Kirchhoff–Kharlamov [3], Clebsch [13], Steklov [2] and Lyapunov [14] solutions, while in structure it does not satisfy the well-known special solutions [3, 6, 8, 11, 12, 15, 16].

Note that when $B_i = 0$, $C_i = 0$ and $\lambda_1 = 0$, conditions (2.15)–(2.19) reduce the equality $a_2 = a_3$, which cannot be satisfied by virtue of conditions (2.6). Hence, there is no analogue of solution (2.21) in the classical problem.

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